M.Sc. - I (Mathematics) (NEP Pattern) Semester-I

NEP-63 - Major (DSC) Paper-III - Linear Algebra

P. Pages : 2 **GUG/S/25/15114**

Time : Three Hours *27558* Max. Marks : 80

Notes : 1. Solve all the **five** questions.
2. All questions carry equal marks.

UNIT – I

1. a) Let W_1 and W_2 be finite dimensional subspace of a vector space V over K then prove that $\dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim(W_1 \cap W_2)$. **8**

b) Let V and V' be finite dimensional vector spaces over K. Then prove that $V \approx V'$ if and only if dim $V = \dim V'$. **8**

OR

c) Let $V_1, \ldots, V_m$ be vector space over a field K. Then prove that $V = V_1 \oplus \ldots\ldots \oplus V_m$ is finite dimensional if and only if each V_i is finite dimensional. In this case, $\dim V_1 \oplus \cdots\cdots \oplus V_m = \dim V_1 + \cdots\cdots + \dim V_m$. **8**

d) If V is a finite dimensional vector space over K then prove that the mapping $e : V \to V^{**}$ defined by $v \to e_v$ is an isomorphism. **8**

UNIT – II

2. a) Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 4 & 2 & 2 \\ 3 & 3 & 2 \\ -3 & -1 & 0 \end{bmatrix}$ over C. **8**

b) Let V be a finite dimensional vector space over K of dimension n. Let the minimal polynomial of T be $m_T(x) = p_1(x)^{r_1} \cdots p_k(x)^{r_k}$, where each $p_i(x)$ is monic irreducible polynomial of degree m_i and let dim ker $p_i(T)^{r_i} = n_i$. Then prove that m_i divides n_i and the characteristic polynomial $C_T(x) = p_1(x)^{n_1/m_1} \cdots p_k(x)^{n_k/m_k}$. In particular, $m_T(x)$ divides $c_T(x)$. **8**

OR

c) Prove that two diagonalizable linear operators S and T on V are simultaneously diagonalizable if and only if they commute i.e. ST = TS. **8**

d) Prove that the geometric multiplicity of an eigenvalue of a linear operator cannot exceed its algebraic multiplicity. **8**

UNIT – III

3. a) Let V be an inner product space over F and $u, v \in V$. Prove that- **8**

i) $\|u \pm v\|^2 = \|u\|^2 + 2\,\mathrm{Re}(u,v) + \|v\|^2$, where Re z denotes the real part of the complex number z.

ii) $\|u+v\|^2 + \|u-v\|^2 = 2\|u\|^2 + 2\|v\|^2$

iii) $\|\lambda u\| = |\lambda|\,\|u\|$, for all $\lambda \in F$.

iv) $|(u,v)| \le \|u\|\,\|v\|$

b) State and prove the Gram-Schmidt orthonormalization theorem. **8**

OR

c) Let V and W be finite dimensional inner product spaces and let $T \in L(V, W)$. **8**

There prove that there exists a unique linear mapping $T^* : W \to V$ such that for all $v \in V$ and $w \in W : (Tv, w) = (v, T * w)$.

d) State and prove Spectral theorem. **8**

UNIT – IV

4. a) Prove that if a bilinear form is reflexive then it is either symmetric or alternating. **8**

b) Let ϕ be a bilinear form on K^n. Then prove that there exists a unique matrix A such that $\phi = \theta_A$. **8**

OR

c) Let ϕ be a reflexive bilinear form on a finite dimensional vector space V over K. Then prove that ϕ is non-degenerate if and only if the matrix of ϕ with respect to an ordered basis of V is invertible. **8**

d) Prove that: Let Char $K \ne 2$ and let $\lambda_1, ..., \lambda_{n-1}, \mu_1, ... \mu_{n-1} \in K$ and $\lambda \in K \setminus \{0\}$. If diag $\{\lambda, \lambda_1, ..., \lambda_{n-1}\}$ and diag $\{\lambda, \mu_1, ..., \mu_{n-1}\}$ are congruent matrices, then so are the matrices diag $\{\lambda_1, ..., \lambda_{n-1}\}$ and diag $\{\mu_1, ..., \mu_{n-1}\}$. **8**

5. a) Let V be a vector space over K. Prove that a finite subset B of V is a basis of V if only if every element of V is unique linear combination of elements of B. **4**

b) Prove that a Jordan chain consists of linearly independent vectors. **4**

c) Prove that: An orthonormal set of non-zero vectors is linearly independent. **4**

d) Define: **4**

i) Bilinear form ii) Bilinear Space

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